Linear Algebra Review Sheet

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1 Introduction

Linear Algebra for many will be a new mathematical language to learn. It is beautiful but it is not intuitive at first. This sheet is meant to highlight the concepts that I struggled with during my first course in Linear Algebra in hopes of explaining them in a way that my previous self would have greatly benefited from. Many of these are from my notes that I took in Math 355 at UWW.

2 Terms, Definitions, Key Introductory Concepts

- Linear Equation: An equation where the highest power of a variable is 1. Geometrically they represent lines, planes, and hyperplanes.
- Linear System: A finite collection of Linear Equations. A solution to a linear system is a solution to all of the equations in the system.
- Affine Function: A function $f : \mathbb{K}^m \to \mathbb{K}^n$ if there exists a vector $\bar{b} \in \mathbb{K}^n$ and a matrix $A \in \mathbb{K}^{m\times n}$ such that: $\forall \bar{x} \in \mathbb{K}^m, f(\bar{x}) = A\bar{x} + \bar{b}$
- Vector: A quantity that must be expressed by more than one scalar.
- Matrix: A matrix is an array of a specific $m \times n$ size that contains indices related to the field in which it is contained.
- Linear Combination: An expression constructed from a set of terms by multiplying each term by a constant and adding the results.
- Consistent: A system has a solution
- Inconsistent: A system does not have a solution.

Concepts

- There are only 3 possible cases when it comes to solutions of linear systems. 0 solutions (all of the linear equations form parallel lines), 1 solution (All equations intersect at a point), and infinite solutions (The equations intersect at a line/hyperplane).

Parametric Form

Suppose we have the REF augmented matrix: $\begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ Let us rewrite this matrix in terms of its variables. $\begin{pmatrix} x + 2y = 3 \\ y = y \\ z = 2 \end{pmatrix}$ We see that x is a function of the free variable y. We can rewrite this in parametric form:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 - 2y \\ y \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} + y \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}$$

3 Invertible Matrix Theorem

For an $n \times n$ matrix A, the following are equivalent:

1. A is invertible.
2. $\det(A) \neq 0$
3. $rank(A) = n$
4. $Ax = b$ has a unique solution $\forall b \in \mathbb{R}^n$.
5. $Ax = 0$ has only the trivial solution $x = 0$.
6. The columns of A are linearly independent.
7. The rows of A are linearly independent.
8. The columns of A span $\mathbb{R}^n$: The column space of A is $\mathbb{R}^n$.
9. The rows of A span $\mathbb{R}^n$: The row space of A is $\mathbb{R}^n$.
10. A is row-equivalent to the identity matrix I_n: You can row reduce $A \to I_n$.
11. The eigenvalues of A are all nonzero.
12. The matrix A^T is invertible.
13. The null space of A is trivial: $Null(A) = \{0\}$.
14. The matrix A has full column rank and full row rank.

3.1 Additional Information

Assume A and B are invertible $n \times n$ matrices for the following.

- A product of an invertible matrix is invertible.
- $(AB)^{-1}AB = I_n$
- $(AB)^{-1}ABB^{-1} = B^{-1}$

4 Determinant

The Determinant is a single number that tells us if A is invertible. If $\det(A) \neq 0$, A is invertible, and thus has a solution. Given a matrix A, its determinant is the signed n volume of the parallelepiped spanned by its columns.

$$\text{Let A be a } 2 \times 2 \text{ matrix } \begin{bmatrix} a & c \\ b & d \end{bmatrix}, \det[A] = ad - bc$$

The determinant tells us something about the linear transformation that the matrix represents.

- $\det(A) > 0$ tells us that the matrix scales the area formed by the vectors by some factor equal to $\det(A)$.
- $\det(A) = 0$ tells us the transformation takes a vector into a lower dimension.
- If $\det(A) < 0$ The transformation is a sort of reflection.

There are some other important qualities of a determinant:

- Swapping two rows or columns changes the sign of the determinant.
- Multiplying an individual row or column by some scalar λ means $\lambda \det(A)$
- Adding rows or columns to each other has no effect on the determinant.

5 Vector Spaces

A vector space is a set whose elements (vectors) can be added together, and multiplied by scalars, and the vector space must contain the $\bar{0}$ vector.
S is a subspace of $\mathbb{K}^n$; a collection of vectors spans S if $span(v_1, v_2, v_3 ... v_k) = S$. This is called the spanning set. A collection of vectors is a spanning set for S if given and vector $w \in S : c_1 v_1 + ... c_k v_k = w$ has at least one solution.

Definition: Basis, a collection of linearly independent vectors that span a vector space.

Given a subspace of S, a basis for S is a collection of vectors that:

- Spans S (There is a linear combination that can hit every point in S)
- Linearly Independent (No vectors in the spanning set of vectors that are linear combinations of other vectors in the spanning set)

A subspace of S must satisfy: $\bar{u}, \bar{v} \in S$

- $\bar{u} + \bar{v} \in S$
- $\bar{0} \in S$

- $\lambda \bar{v} \in S$

Every vector space has a basis and all the bases of a vector space have the same size. The dimension of the basis defines the size of a vector space. The dimension is simply the number of vectors in the basis.

Vectors $\{v_1, v_2, v_3 ... v_n\}$ are linearly independent iff $c_1 v_1 + c_2 v_2 + c_3 v_3 + .. + c_n v_n = 0$ has only the trivial solution of $c_1 = c_2 = c_3 = 0$.

5.1 $\mathbb{P}$ Vector Spaces

We denote the vector space of all n-degree polynomials or less as $\mathbb{P}^n$.

$$\mathbb{P}^n = \{p(x) | p(x) = \sum_{k=0}^{n} a_k x^k, a_k \in \mathbb{R}/\mathbb{C}\}$$

A basis for $\mathbb{P}$ is the set of monomials

$$\{1, x, x^2, ..., x^n\}$$

. Any polynomial can be described as a unique combination of these elements. Note that the dimension of $\mathbb{P}^n$ is $n+1$ since the indexing starts at 0.

Since we can add polynomials and multiply them by scalars and never leave the vector space, it is closed under vector addition and scalar multiplication. A subset of $\mathbb{P}$ must also be closed under these conditions such that it never leaves that subset.

A vector $\bar{v} \in \mathbb{P}^n = \begin{pmatrix} a_0 \\ a_1 \\ a_2 \\ . \\ . \\ . \\ a_n \end{pmatrix}$ where a_i is the coefficient of x^i.

6 Linear Transformations

A linear transformation between two vector spaces is a function T that satisfies:

- $T(\bar{v} + \bar{w}) = T(\bar{v}) + T(\bar{w})$
- $T(\lambda \bar{v}) = \lambda T(\bar{v})$
- $\lambda = 0 \rightarrow T(\bar{0}) = \bar{0}$

Theorem: A function from a finite-dimensional vector space V to a finite-dimensional space W is a linear transformation iff there is a matrix such that $T(\bar{x}) = A\bar{x}$. A is called the standard matrix for $[T]$. Its columns represent what happens to the standard basis vectors. The i column of A represents the transformation $T(e_i)$.

Let A be an $m \times n$ matrix representing a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$.

- Range of A is the subspace of $\mathbb{R}^m$ spanned by the vectors of the form Ax, where $x \in \mathbb{R}^n$.
- The null space (kernel) of A is the subspace of $\mathbb{R}^n$ consisting of all the vectors x satisfying $Ax = 0$.

Steps to Find the Range:

Suppose we have an invertible matrix $A_{2\times 2}$,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The range of A is the span of its columns with pivots in REF. Since A is invertible it follows that A has pivots in every column, we write $span\{\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix}\}$

To find the null space, place the matrix into RREF, and express the pivots in terms of the free variables. These vectors form the null space. Since our matrix A is invertible, it only contains the trivial solution $\{0\}$.

$$rank(A) = dim(Range(A)) \ \& \ nullity(A) = dim(Null(A))$$

In other words, rank is the number of pivots in REF. Nullity is the number of free variables in REF.

Let us work through an example:

Let $T \to \mathbb{R}^3 \to \mathbb{R}^2$ given by:

$$T(\begin{bmatrix} x \\ y \\ z \end{bmatrix}) = \begin{bmatrix} x + y \\ y + z \end{bmatrix}$$

Let us write out this matrix in terms of its linear equations

- $1x + 1y + 0z$
- $0x + 1y + 1z$

We can put the coefficients into a matrix like so:

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

Our first step is to get T in REF; in this case, we can see that there are pivots in the first and second columns and thus we can determine the range of T is the span of those columns.

$$span(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix})$$

To solve for the kernel (null) we solve using the equation:

$$A\bar{x} = \bar{0}$$

Keeping in mind that $x \in \mathbb{R}^n$ and $\bar{0} \in \mathbb{R}^m$ so in this case, $x \in \mathbb{R}^3$ and $0 \in \mathbb{R}^2$

Or to put it simply, since we have a transformation from $\mathbb{R}^3 \to \mathbb{R}^2$ we will write,

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Row reducing the augmented matrix into RREF gives us

$$\begin{bmatrix} 1 & 0 & -1 & | & 0 \\ 0 & 1 & 1 & | & 0 \end{bmatrix}$$

Now we can write our pivot variables in terms of their free variables

- $x_1 - x_3 = 0$
- $x_2 + x_3 = 0$
- x_3 is free so we write $x_3 = x_3$

Which, of course, simplifies to:

- $x_1 = x_3$
- $x_2 = -x_3$
- $x_3 = x_3$

Thus, our solution is:

$$x = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

And so, the null space of our matrix is:

$$Null(A) = span \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

Fundamental Theorem of Linear Algebra

$rank + nullity =$ number of columns.

7 Linearly Independent Sets - Bases

Consider the elementary vectors of $\mathbb{R}^2$, e_1 and e_2. The span of these vectors can be written as:

$$span(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix})$$

To determine if a set of vectors spans a vector space, we check if the spanning set is invertible by row reducing and checking if the pivots match the dimension, or by taking the determinant. If it is invertible, and the dimension of the spanning set is the same as the dimension of the vector space.

Let's say: $B = \{b_1 b_2\}$, where $b_1 = \begin{bmatrix}1\\1\end{bmatrix}$ and $b_2 = \begin{bmatrix}2\\-1\end{bmatrix}$

Thus, the B-basis matrix is $B = \begin{bmatrix}1 & 2\\1 & -1\end{bmatrix}$. We also have a vector in standard coordinates that we wish to express with respect to our basis B, $v = \begin{bmatrix}3\\1\end{bmatrix}$. We must solve

$$Bc = v$$

Where again, B is the matrix with our basis vectors as columns and $c = \begin{bmatrix}c_1\\c_2\end{bmatrix}$ is the coordinate vector in B basis. So in order to find c,

$$\begin{bmatrix}1 & 2\\1 & -1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$

We can solve this in several ways. We can row reduce or simply find the inverse of B since:

$$c = B^{-1}v$$

In solving by finding the inverse we see,

$$c = B^{-1}v = \begin{bmatrix}\frac{1}{3} & \frac{2}{3}\\\frac{1}{3} & -\frac{1}{3}\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}\frac{5}{3}\\\frac{2}{3}\end{bmatrix}$$

So, our vector v, in base B, is $\begin{bmatrix}\frac{5}{3}\\\frac{2}{3}\end{bmatrix}$.

Standard Basis → B-Basis

$$v_B = B^{-1}v_{std}$$

B-Basis → Standard Basis

$$v_{std} = Bv_B$$

B-Basis → C-Basis

To express a vector v_B in the C-Basis, convert v_B to the standard basis:

$$v_{std} = Bv_B$$

Then convert v from the standard basis to the C-Basis.

$$v_C = C^{-1}v_{std}$$

So the combined formula is:

$$v_C = C^{-1}Bv_B$$

Change of Basis Matrix for B → C

$$P_{B\to C} = C^{-1}B$$

$$v_C = P_{B\to C}v_B$$

Let us do another example as this can be quite challenging:
Suppose we have the vectors v_1 and v_2 such that

$$v_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, v_2 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

These vectors form a basis which we call δ. If we take the vectors of δ and form a column matrix, which we will call Δ, we get an essential matrix:

$$\Delta = [v_1, v_2] = \begin{bmatrix} 1 & 3 \\ 2 & 3 \end{bmatrix}$$

This matrix converts (through matrix multiplication) any vector written with respect to δ, into the standard basis coordinates, which as a reminder is:

$$E_{std} = [e_1, e_2] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This comes from the formula mentioned earlier:

$$u_{std} = \Delta u_\delta$$

Where:

- u_δ is the vector of coordinates in the δ basis
- u_{std} is the vector expressed in the standard basis.

What if you want to convert a vector from u_{std} to u_δ? Our goal should be to look at the previous equation and try to isolate our u_δ. We can do this by taking the inverse of both sides,

$$\Delta^{-1}u_{std} = \Delta^{-1}\Delta u_\delta$$

Remember that $\Delta^{-1}\Delta = I_n$ thus,

$$\Delta^{-1}u_{std} = u_\delta$$

8 Eigenvalues, Eigenvectors, and Diagonality

We want to find when a matrix is diagonal up to a change of basis.

Let A be an $n \times n$ matrix and $\bar{v}$ is a nonzero vector and λ is a scalar. We say λ is an eigenvalue of A and $\bar{v}$ is an eigenvector corresponding to λ if:

$$A\bar{v} = \lambda\bar{v}$$

Steps to find Eigenvalues, Eigenvectors, and Diagonalize:

First start by taking the $\det[A - \lambda I_n]$, this yields what is called the characteristic polynomial. Find the solutions to this polynomial. These solutions are the eigenvalues.

We can find the eigenvectors by looking at the null space. We do this by solving $(A - \lambda I_n)\bar{v} = \bar{0}$ for each eigenvector. This tells us that the eigenspace with respect to λ is $E_\lambda(A) = Null(A - \lambda I_n)$.

The diagonalization of a matrix is simply the eigenvalues along the diagonal, for a 2×2 matrix A this will look like $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2. \end{bmatrix}$

Multiplicities

The algebraic multiplicity of an eigenvalue λ is the multiplicity of it as a root of the characteristic polynomial.

The geometric multiplicity is the dimension of the eigenspace.

$1 \leq$ Geometric Multiplicity $\leq$ Algebraic Multiplicity

It may be useful when solving the characteristic polynomial to use the rational roots theorem. The rational root theorem tells us that if the leading coefficient a_n and trailing coefficient a_0 of a polynomial are nonzero integers, then we can solve for the roots by checking the positive and negative factors of a_0 and a_n.

8.1 Diagonalization

A matrix is only diagonalizable iff all geometric multiplicities are equal to their algebraic multiplicities. A square matrix A is diagonalizable if it can be expressed as:

$$A = PDP^{-1}$$

Where P is a matrix whose columns are the eigenvectors of A, and D, is a diagonal matrix whose entries are the eigenvalues of A. We can use this formula to find D as well,

$$P^{-1}AP = D$$

9 Important Matrices and Vectors

9.1 Transformations

Rotation Matrix (CCW)

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

9.2 Other

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$